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Fisher renormalisation and the application of RSRG methods to annealed dilute systems

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Abstract. The application of RSRG methods to annealed random systems is studied with special reference to the violation of Fisher renormalisation by approximate techniques. An RSRG cluster approximation is first applied to the annealed dilute Ising model to obtain flowlines and fixed points. Other possible forms of scaling are considered, and how they may be tested against the scaling of equivalent pure models. It is argued that to satisfy Fisher renormalisation it is necessary to allow the probability distribution of the annealed system to evolve, under length scaling, from its initial simple form. The evolution is investigated for the annealed dilute q -state Potts model on the diamond Berker lattice (which can have either sign for the pure specific heat exponent α). By using the width of the distribution as a crucial additional scaling parameter, it is shown how the flow in the increased parameter space can satisfy the Fisher renormalisation criterion and allow the proper appearance of the random fixed point. The treatment is less complete than that for the quenched case because of the need to approximate a relation between indicator functions and the probability distribution.

1. Introduction

Real space renormalisation group (RSRG) methods (see Burkhardt and van Leeuwen 1982) have been extensively applied to lattice based critical phenomena, since they incorporate the crucial scaling concept in a very direct way allowing contact with intuitive physical ideas. The RSRG method is particularly appropriate for critical properties of systems with randomness such as dilution since the position space formulation can deal directly with strong local disorder. The method, however, has its limitations, of which the best known and most important is the uncontrolled nature of its approximations such as the cluster truncation normally used. The application to random systems, however, introduces a new source of approximation, related to the scaling of the distribution describing the randomness. It has been discussed elsewhere (da Cruz and Stinchcombe 1986, hereafter called ACS) how this approximation can lead to violation of the well known Harris criterion (Harris 1974) for quenched random systems, and how this difficulty can be resolved. In the present paper we consider the analogous problems for annealed random systems.

In the annealed dilute problem the disorder variable is also a thermodynamic variable (see Stinchcombe 1983). In simple bond dilution it can be modelled by an extra variable in the middle of the bond (decoration of the lattice: see Syozi and Miyazima (1966) and Syozi (1972)). One important aspect of bond-decorated lattices is that, because the decorating variable interacts only with its two neighbouring sites,

its local environment is like a 1D chain. Consequently one can trace out the decorating variables to convert the problem into another involving only the spin variables interacting through a nearest-neighbour pair interaction on the matrix lattice. For the annealed system this maps the problem into an equivalent pure one, so that the exact result is known whenever the corresponding pure problem can be exactly solved. This was first realised by Syozi (1972) and Rapaport (1972). Thorpe and Beeman (1976) treated the annealed problem with an arbitrary distribution $P(J)$ of the exchange interaction J , in the same fashion.

The extra thermodynamic variable in the annealed case can produce a critical behaviour different from the pure one. This change of behaviour will naturally be reflected in the critical exponents. Its renormalisation has been studied in a more general context by Essam and Garelick (1967) and Fisher (1968) and became known as *Fisher renormalisation*. For annealed magnets the Fisher renormalisation states that if the pure specific heat exponent α is positive the annealed critical exponents take 'renormalised' values given by

$$\begin{aligned}
 \alpha_{\text{pure}} &\rightarrow \tilde{\alpha} = -\alpha_{\text{pure}}/(1 - \alpha_{\text{pure}}) \\
 \beta_{\text{pure}} &\rightarrow |\tilde{\beta}| = \beta_{\text{pure}}/(1 - \alpha_{\text{pure}}) \\
 \gamma_{\text{pure}} &\rightarrow |\tilde{\gamma}| = \gamma_{\text{pure}}/(1 - \alpha_{\text{pure}}) \\
 \delta_{\text{pure}} &\rightarrow |\tilde{\delta}| = \delta_{\text{pure}} \\
 \nu_{\text{pure}} &\rightarrow \tilde{\nu} = \nu_{\text{pure}}/(1 - \alpha_{\text{pure}}) \\
 \eta_{\text{pure}} &\rightarrow \tilde{\eta} = \eta_{\text{pure}}
 \end{aligned} \tag{1}$$

where the exponents on the right-hand side are the pure ones. If α_{pure} is negative, however, the annealed exponents are the same as the pure ones. Therefore $\tilde{\alpha}$ is always negative and consequently the specific heat is not divergent. These renormalised exponents satisfy the usual scaling relations.

The criterion, $\alpha_{\text{pure}} > 0$, for Fisher renormalisation is the same as the Harris criterion for crossover to new ('random') critical behaviour in systems with quenched disorder. Approximate techniques usually violate such a criterion and, *a fortiori*, the exact relations (1).

An RSRG method is given for the annealed dilute Ising problem in § 2 (where preliminaries for later discussions are also introduced, including the mapping to the pure system, and a generalised hyperscaling relation). Though such RSRG methods have many attractions it is unsatisfactory that they do not automatically satisfy the Fisher renormalisation criterion or the relations (1). The origin of this inadequacy is discussed in § 3 and (as in ACS for the quenched system) restricting the evolution of the distribution function is suggested as the cause. In § 3 various scaling approaches are outlined, some exploiting the relationship of the annealed system to the equivalent pure system. One of these different possibilities provides a 'trick' way of exactly satisfying Fisher renormalisation within an RSRG approach. Others suggest ways of testing where the violation of the Fisher criterion arises in more standard RSRG approaches. Sections 4 and 5 carry out such a test, on the annealed diluted q -state Potts model on a Berker lattice: this system is considered because its pure limit can have either sign of α . The analytic preparations are given in § 4, and the results are obtained in § 5 by continuing with a numerical attack, which confirms the need to allow the evolution of the distribution under scaling. Section 6 is a concluding discussion.

2. Use of RSRG on annealed dilution

The RSRG method for annealed dilute problems is illustrated here by an application to the Ising system. The annealed bond-dilute Ising square net can be treated by an RSRG cluster decimation or Migdal method. In the simplest form of cluster decimation the trace is taken over two ('decimated') spins of a four-spin cluster to scale its Gibbs factor into that for a single renormalised bond (length scale factor $b = \sqrt{2}$):

$$\exp(K' S_i S_j \eta_{ij} + \mu' \eta_{ij}). \quad (2)$$

Here K' and μ' are the scaled values (indicated by primes) of exchange and chemical potential (times a factor β) and η is the disorder variable that indicates the absence or presence of the bond by assuming values 0 or 1. This gives

$$t' = \frac{2(1+X)^2 t^2}{[2+4X+X^2(1+t^4)]} \quad (3)$$

$$X' = \frac{X[2+4X+X^2(1+t^4)]}{(1+2X)^2} \quad (4)$$

where $t \equiv \tanh K$ and $X \equiv e^\mu \cosh K$. Exactly the same equations (3) and (4) also result in the simplest form of Migdal approximation. However here the form of cluster is different and the length scaling corresponds to $b = 2$.

The bond concentration is given by the thermal average $\langle \eta \rangle$, and μ is temperature dependent to keep p fixed. $\langle \eta \rangle$ is most easily evaluated by using the mapping (given hereafter) to a pure system by means of a partial trace over the disorder variables.

The grand partition function Ξ of the annealed system has the trace over both thermodynamic variables

$$\Xi = \text{Tr}_{\{S\}} \text{Tr}_{\{\eta\}} e^{-\beta H}. \quad (5)$$

The trace over η , however, can be taken separately as each η only involves a single bond. The reduction for each bond is

$$\text{Tr}_{\eta_a} \exp(K S_1 S_2 \eta_a + \mu \eta_a) = \exp(\tilde{K}_0 + \tilde{K} S_1 S_2) \quad (6)$$

with

$$\exp(2\tilde{K}_0) = [\exp(\mu + K) + 1][\exp(\mu - K) + 1] \quad (7)$$

$$\exp(2\tilde{K}) = [\exp(\mu + K) + 1]/[\exp(\mu - K) + 1]. \quad (8)$$

This result expresses the original annealed system in terms of an equivalent pure Ising model on the same lattice, with variables \tilde{K} , \tilde{K}_0 .

By letting $\mu \rightarrow \infty$, in this last equation, one finds that $\tilde{K} \rightarrow K$ so that the pure limit ($p = 1$) of the original system occurs when $X = \infty$.

Using these results the grand partition function of the annealed system can be rewritten as

$$\Xi = \exp(N\tilde{K}_0) Z(\tilde{K}) \quad (9)$$

where N is the number of bonds and $Z(\tilde{K})$ is the partition function of the equivalent pure system. This gives

$$\langle \eta \rangle = p = \frac{1}{N} \frac{\partial \ln \Xi}{\partial \mu} = \frac{\partial \tilde{K}_0}{\partial \mu} + \frac{1}{N} \frac{d \ln Z(\tilde{K})}{d \tilde{K}} \frac{\partial \tilde{K}}{\partial \mu}. \quad (10)$$

The derivatives of \tilde{K}_0 and \tilde{K} with respect to μ can be calculated exactly from (7) and (8). The factor that multiplies this last derivative is the nearest-neighbour spin-spin correlation function $\langle S_1 S_2 \rangle$, which in this system is also the dimensionless internal energy per bond. Of course, this can be calculated exactly (see McCoy and Wu 1973). For consistency, however, it will here be calculated by applying RSRG to the equivalent system using the same four-bond cluster that leads to (3) and (4).

Defining the dimensionless free energy per bond as

$$\tilde{f} \equiv \beta F / N = -\ln Z / N \tag{11}$$

the required factor can be expressed as $-d\tilde{f}(\tilde{K})/d\tilde{K}$.

From Nauenberg and Nienhuis (1974) decimation on the equivalent pure system yields

$$\tilde{f} = -\lim_{n \rightarrow \infty} \sum_{m=0}^n \frac{g[\tilde{K}^{(m)}]}{L^m} \tag{12}$$

where L is the ratio of the number N of bonds in the original cluster to that in the decimated cluster and g is given by

$$\exp(H[\tilde{K}'] + Ng[\tilde{K}]) = \text{Tr}_{\{s\}} \exp(H[\tilde{K}]) \tag{13}$$

where the trace is, as usual, over the decimated spins.

The explicit form of (13) for the present case is (see Jayaprakash *et al* 1978)

$$\exp(\tilde{K}' S_1 S_4 + 4g(\tilde{K})) = \text{Tr}_{S_2, S_3} [\exp(\tilde{K} S_1 S_2 + \tilde{K} S_2 S_3 + \tilde{K} S_1 S_3 + \tilde{K} S_3 S_4)] \tag{14}$$

giving

$$g[\tilde{K}^{(m)}] = \frac{1}{4} \ln(4 \cosh 2\tilde{K}^{(m)}) \tag{15}$$

and

$$\tilde{K}^{(m)} = \ln(\cosh 2\tilde{K}^{(m-1)}) \tag{16}$$

where the value used for L was four (since four bonds of the original cluster are replaced by a single renormalised bond). This then gives

$$-\frac{df(\tilde{K})}{d\tilde{K}} = \lim_{n \rightarrow \infty} \sum_{m=0}^n \frac{1}{4^m} \frac{dg[\tilde{K}^{(m)}]}{d\tilde{K}} \tag{17}$$

Using the chain rule and (15) the derivative on the RHS of the last equation can be written as

$$\frac{dg[\tilde{K}^{(m)}]}{d\tilde{K}} = \frac{1}{4} 2^m \prod_{l=m}^0 \tanh[2\tilde{K}^{(l)}] \tag{18}$$

Thus (10), (17) and (18) give p in terms of \tilde{K} , \tilde{K}_0 which are, in turn, known from (7) and (8) in terms of μ , K , or equivalently X , t . The scaling of these latter variables, given in (3) and (4), can then be converted to scaling of p , t , and this procedure results in the flowlines plotted in figure 1. The arrows indicate the direction of flow and their spacing the speed of flow. The fixed points are represented by semi- and quarter circles. The bold lines are the critical curves. Phase 1 has a non-zero magnetisation and an infinite cluster is present. Phase 2 has no net magnetisation, although the infinite cluster is still present. Phase 3 has no infinite cluster and therefore no net

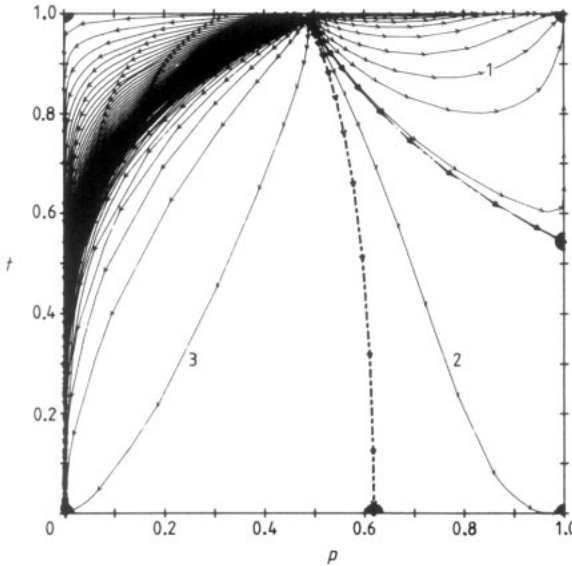


Figure 1. Flowlines of the annealed dilute Ising model on the 2D square net are plotted in the space of parameters p = concentration, $t = \tanh J/K_B T$. The arrows indicate the direction of flow and their spacing the speed of flow. The fixed points are represented by semi- and quarter circles. The thick lines are the critical curves. Phase 1 has a non-zero magnetisation and an infinite cluster is present. Phase 2 has no net magnetisation, although the infinite cluster is still present. Phase 3 has no infinite cluster and therefore no net magnetisation.

magnetisation. The bold broken curve (separating phases 2 and 3) is not a straight line, as it would be in a quenched system. This is due to the thermal dependence of the probability of a bond being present.

The question now arises whether figure 1 is consistent with the Fisher criterion. On the critical line between phases 1 and 2 the flow is towards the pure fixed point. This is consistent if $\alpha_{\text{pure}} < 0$. However from (7) we can calculate ν_{pure} and obtain α_{pure} using hyperscaling. For the cluster decimation interpretation, where $b = \sqrt{2}$, ν_{pure} turns out to be such that inserting $d = 2$ into hyperscaling leads to $\alpha_{\text{pure}} > 0$, thus implying a violation of the Fisher criterion. On the other hand, the Migdal cluster interpretation, where $b = 2$, leads, with $d = 2$, to $\alpha_{\text{pure}} < 0$, implying no violation. This ambiguous result can be clarified by the following generalised view. The pure specific heat exponent can be evaluated directly from (17), and it turns out to be negative and equal to

$$\alpha_{\text{pure}} = 2 - d_f \nu_{\text{pure}} \tag{19}$$

where $\nu_{\text{pure}} = \ln b / \ln \lambda$, $d_f \equiv \ln L / \ln b$ with λ the eigenvalue of the (pure) transformation (16) linearised about its critical fixed point. d_f is the fractal dimension of the system as represented by the cluster approximation being used, and this is two, the same as d , for the Migdal interpretation, but is four for the cluster interpretation. If the proper fractal dimension is used, the (generalised) hyperscaling relation (19) and the Fisher criterion are satisfied. These results are shown in table 1.

Unfortunately this does not dispose of a much more basic and general problem: Fisher renormalisation (i.e. the relationships (1)) and even the Fisher criterion can be violated because the indicator variables change their nature under scaling. This has

not been allowed for in the discussion of this section and is the topic treated in the remaining sections.

3. Problems and possibilities in the scaling of annealed systems

When annealed dilute systems are scaled the indicator variables change their form. This is because when a set of indicator (or 'bond') variables, each taking two values, is replaced (under decimation) by a single variable, that variable has strictly more than two outcomes. This generalisation is like the evolution of the (originally binary) bond distribution function occurring in the scaling of quenched dilute systems. In most RSRG treatments of quenched dilute systems, this effect is ignored and, as shown in dcs, it can lead to the violation of the Harris criterion. Similarly, the treatment of annealed dilution given in the previous section suppresses the evolution of the distribution of outcomes for the indicator variable η (which was taken to remain binary in (2)). We now investigate the consequences of this approximation for the annealed system and its generalisation.

As annealed systems can be mapped into an equivalent pure one, one should be able to assess the effect of the binary approximation by studying the behaviour of an annealed system and its pure equivalent under length scaling.

What can be done is to study three systems in the manner shown in figure 2. In the first system, the dilute annealed problem is mapped into a pure one and decimation is applied to the result to obtain \tilde{K}'_p . Here the tilde means that the system has been mapped into pure and the prime that it has been scaled once. In the second case the dilute system is first scaled, allowing for the full proliferation of bond values, and only then mapped into pure to obtain \tilde{K}'_a . In the last one, scaling is done using the binary approximation (as above) and the result is mapped into pure to find \tilde{K}_b . From the analytical work given in da Cruz (1985) it can be shown that all these results agree. If in the process described above, however, instead of performing a single scaling, two are done, the result obtained using the binary approximation starts to diverge from the others (see again da Cruz (1985)). This confirms the need to allow for the

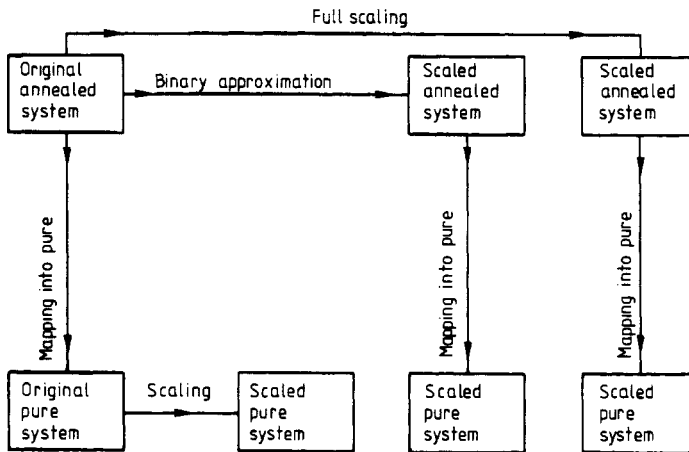


Figure 2. Diagram of the proposed calculations. The three scaled pure results are to be compared.

proliferation of bond values and, as argued in *acs* for the quenched case, we expect for the annealed case also that a broadening of the distribution of bond values will be the crucial signature of (and requirement for) an instability of the pure behaviour to the randomness and hence to Fisher renormalisation. This is confirmed in the results of § 5.

An alternative ‘trick’ way (not pursued further here) of dealing with the annealed problem, without the need to allow for the full proliferation of bonds and still not compromising the results by using the binary approximation, can be seen from figure 2. One can map the system into pure, scale it and then map it back into annealed. This last mapping is not unique in itself, but can be made so by an additional statement such as that p remains constant.

4. Analytical calculation

We have argued, following *acs* for the quenched case, that a fuller account of the form of the distribution is needed (though, as mentioned at the end of the previous section, it is possible to avoid this by performing the scaling in the equivalent pure system).

Because of the nature of the annealed problem the treatment of the distribution is not as straightforward as in the quenched case. Nevertheless it will be shown to be possible in the following section and subsequently results similar to the ones obtained in *acs* will be obtained.

To perform the calculation proposed in the last paragraph but one of § 3 it is necessary first to generalise the Hamiltonian for the problem in two ways. The first generalisation is to accommodate the different possible bond values. Following Thorpe and Beeman (1976) the Gibbs factor for a single bond of the Ising model can be written as

$$\exp(-\beta H_{12}) = \exp\left(\sum_i K_i f_i S_1 S_2 + \sum_i \mu_i f_i\right) \tag{20}$$

where $K_i = J_i/k_B T$, $\mu_i = \xi_i/k_B T$ (ξ_i is a chemical potential) and the thermodynamic variable f_i is an indicator function (see Moran 1968) that is one if that bond has an exchange J_i and zero otherwise. Of course ξ_i and J_i are associated so as to make $\langle f_i \rangle$ temperature independent and equal to the concentration of that particular bond type. The same ideas used in the previous section apply here and the mapping to the pure system is (see (6)-(8))

$$\text{Tr}_{f_i} \exp\left(\sum_i K_i f_i S_1 S_2 + \sum_i \mu_i f_i\right) = \sum_i \exp(K_i S_1 S_2 + \mu_i) = \exp(\tilde{K}_0 + \tilde{K}_1 S_1 S_2) \tag{21}$$

which gives

$$\exp(2\tilde{K}_0) = \left(\sum_i \exp(\mu_i + K_i)\right) \left(\sum_i \exp(\mu_i - K_i)\right) \tag{22}$$

$$\exp(2\tilde{K}) = \left(\sum_i \exp(\mu_i + K_i)\right) \left(\sum_i \exp(\mu_i - K_i)\right)^{-1} \tag{23}$$

One important point to notice is that the inverse transformation is not uniquely defined.

The second generalisation is to the q -state Potts model on the diamond Berker lattice (in place of the Ising model, which corresponds to $q = 2$). The generalisation to arbitrary q allows both possible signs of α_{pure} . The use of the Berker lattice avoids all approximations in treating the system by RSRG in the pure limit.

Decimation will be performed using the cluster shown in figure 3. For a generic configuration of bonds this gives the following scaling equation:

$$\begin{aligned} &\exp[K'_0 + \mu'_i + K'_i(q\delta_{S_A, S_B} - 1)] \\ &= \exp(4K_0 + \mu_1 + \mu_2 + \mu_3 + \mu_4) \text{Tr}_{S_1}\{\exp[K_1(\delta_{S_A, S_1} - 1) + K_2(\delta_{S_1, S_B} - 1)]\} \\ &\quad \times \text{Tr}_{S_2}\{\exp[K_3(\delta_{S_A, S_2} - 1) + K_4(\delta_{S_2, S_B} - 1)]\}. \end{aligned} \tag{24}$$

The indicator functions are absent because it was assumed that the bonds have the prescribed values. The chemical potentials are present characterising the problem as annealed. Taking the traces we obtain

$$\begin{aligned} &\exp(K'_0 + \mu'_i)v'_i[1 + u'_i(q\delta_{S_A, S_B} - 1)] \\ &= [\exp(K_0 + \mu_1)v_1][\exp(K_0 + \mu_2)v_2][\exp(K_0 + \mu_3)v_3][\exp(K_0 + \mu_4)v_4] \\ &\quad \times q^2\{1 + (u_1u_2 + u_3u_4)(q\delta_{S_A, S_B} - 1) + u_1u_2u_3u_4[q(q - 2)\delta_{S_A, S_B} + 1]\} \end{aligned} \tag{25}$$

where

$$\begin{aligned} v_i &\equiv \exp(-K_i)[\exp(qK_i) + q - 1]/q \\ u_i &\equiv [\exp(qK_i) - 1]/[\exp(qK_i) + q - 1]. \end{aligned}$$

If the spin variables are traced out of this Gibbs factor, the result will give the unnormalised probability of this particular arrangement. Of course this is not the probability for u'_i , as other sets of u_i can also give the same u'_i (due to symmetry, etc). Therefore we find

$$\begin{aligned} \tilde{p}'_i &= q \exp(K'_0 + \mu'_i)v'_i \\ &= \sum [\exp(K_0 + \mu_1)v_1][\exp(K_0 + \mu_2)v_2] \\ &\quad \times [\exp(K_0 + \mu_3)v_3][\exp(K_0 + \mu_4)v_4]q^3[1 + (q - 1)u_1u_2u_3u_4]. \end{aligned} \tag{26}$$

Here the tilde indicates that the probability is unnormalised and the sum is over all arrangements of the u_i that give u'_i . To normalise it is necessary to divide this result by the partition function, which here is the sum of all \tilde{p}'_i . The probability is thermally dependent because of the u_i . The approximate nature of this calculation must be

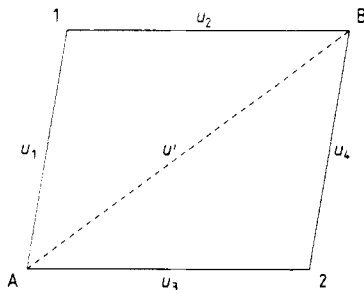


Figure 3. Cluster on which decimation is to be performed. The numbered sites are to be decimated, producing the scaled bond u' .

stressed, as correlations between indicator variables, produced by spins common to different clusters, were neglected.

Writing separately, from (25), the parts that depend on the Kronecker delta and those which do not and dividing the former by the latter, one finds

$$u'_i = \frac{u_1 u_2 + u_3 u_4 + (q-2) u_1 u_2 u_3 u_4}{1 + (q-1) u_1 u_2 u_3 u_4} \quad (27)$$

which is the same as the quenched result (see *acs*). This is not surprising since it is the probability aspect that differentiates the annealed from the quenched and once the value of the bonds is specified the scaled bond should be the same.

From the form of (26), one can easily see how a further scaling can be done. What is needed are the values of the products $v'_i \exp(K'_0 \mu'_i)$ and u'_i . The latter ones come directly from (27) and the former are simply (26) divided by q .

To treat the problem numerically one has to be able to obtain the $v_i \exp(K_0 + \mu_i)$ from the p_i and the u_i , as was described in the above paragraph; however the inverse relation is also needed. In this problem we, in principle, start with a generalised distribution and make it into a histogram, which gives p_i and u_i . We need then to obtain from these $v_i \exp(K_0 + \mu_i)$. We know that this is equal to \tilde{p}_i / q and $p_i = \tilde{p}_i / A$, where A is the sum of all \tilde{p}_i and therefore is unknown. This constant, however, is not important, because it will disappear when the \tilde{p}_i are normalised. Use of the scaling equations (26) and (27) result in a scaled set of variables from which the scaled distribution is obtained.

5. Results of numerical calculation

The calculations outlined above were carried out numerically following the procedure established in *acs*. One particular point is mentioned here: in order to follow those aspects which are related to Fisher renormalisation and the appearance of a random fixed point it is sufficient to use a distribution, all of whose weight is at non-vanishing u (because the weight of missing bonds scales away for concentrations greater than the threshold) and to follow, in particular, the scaling of its mean u and width σ . A Gaussian distribution was actually used and, as in the quenched case investigated in *acs*, this is an adequate approximation to the invariant shape. The flowlines in the space of u and σ are plotted in figures 4–6 for q equal to 1, 6 (for which $\alpha_{\text{pure}} < 0$) and 7 ($\alpha_{\text{pure}} > 0$). In these figures, the full flowlines are the ones which always flow towards smaller values of the width and the broken flowlines are the ones which do not. The arrows in the flowlines indicate direction of flow and its spacing the speed of flow. The asterisk in the abscissa is the pure fixed point and the star is the random fixed point. In agreement with the Fisher criterion, this latter fixed point occurs in our calculations when $q \geq 7$ which is the case in which $\alpha_{\text{pure}} > 0$, and for the other values of q ($\alpha_{\text{pure}} < 0$) the width scales away, showing that randomness is irrelevant. The line going through pure and random fixed points that appears in the plots for $q = 7$ is the critical line.

The random fixed point was found (for $q \geq 7$) by solving the simultaneous equations $\mu' - \mu = 0$ and $\sigma' - \sigma = 0$ by a numerical method which is a combination of bisection and interpolation. The linearised form of the transformation around the random fixed point was obtained by numerical calculation of the differentials that are its matrix

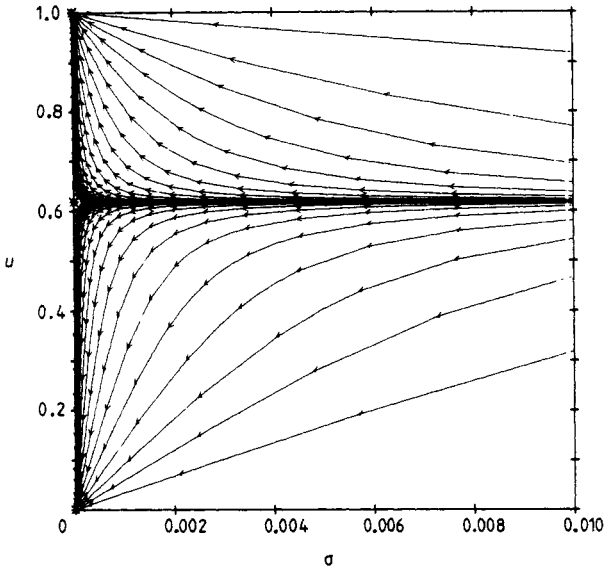


Figure 4. Flowlines for the annealed random $q = 1$ Potts model. The parameters are the mean u and width σ of the distribution of bond variables. The arrows indicate the direction and speed of flow. The flowlines always flow towards smaller value of the width, which is consistent with the criterion for no Fisher renormalisation, as α_{pure} is negative. The asterisk in the ordinate indicates the position of the pure fixed point.

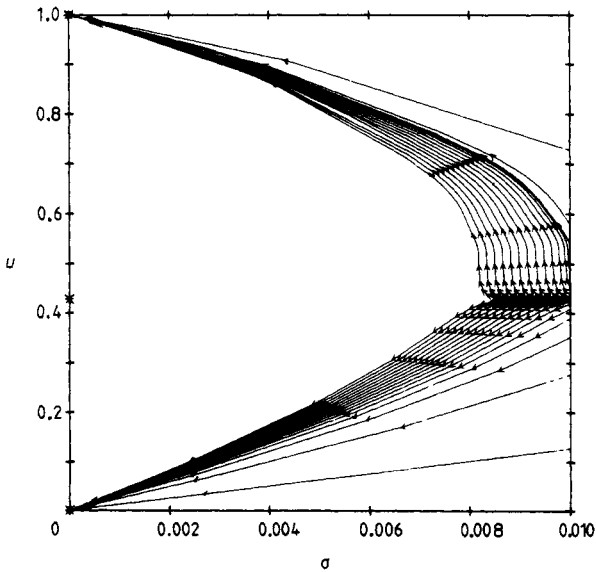


Figure 5. Flowlines for the $q = 6$ Potts model. The arrows indicate the direction and speed of flow. The flowlines always flow towards smaller value of the width (which is consistent with the negative value of α_{pure}). The asterisk in the ordinate indicates the position of the pure fixed point.

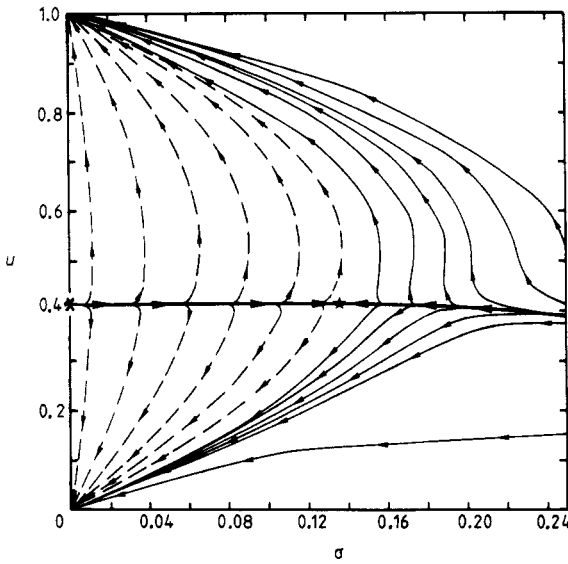


Figure 6. Flowlines for the $q = 7$ Potts model. The labelling and other conventions are as in figure 4. The full flowlines always flow towards smaller value of the width, while the broken ones initially flow toward larger values. The asterisk in the ordinate indicates the position of the pure fixed point and the star indicates the random fixed point. The curve that goes through both fixed points is the critical line.

elements. From these results the eigenvalues were found and from these the ‘random’ critical exponents. These results are presented in table 2 for q from 7–10.

The critical exponents are again presented in table 3, but there they are compared with the predictions of Fisher renormalisation. We can see that they all change in the same direction, i.e. from the pure values towards the Fisher renormalisation predictions, but they are still much closer to the pure values than to the Fisher renormalisation predictions, and it is especially disappointing to see that α is positive. This is discussed in the next section. It is interesting to see that the quenched results (α_{CS}) are very close to the values prescribed by the Fisher renormalisation.

Table 1. Comparison of RSRG exponents for the annealed 2D Ising model with those predicted by Fisher renormalisation applied to RSRG pure and exact results.

Pure fixed point	RSRG result	Exact result	
ν_t	0.67	1	
α_t	-0.66	0	
Percolation fixed point	RSRG result	Fisher renormalisation from exact	Fisher renormalisation from RSRG
ν_p	0.67		
α_p	0.66		
ν_t	1.01	1	0.67
α_t	-0.02	0	-0.66

Table 2. Values at the random fixed point. u^* is the mean of the distribution, σ^* is the width of this distribution, λ_b is the eigenvalue, ν is the correlation length critical exponent and α is the specific heat critical exponent.

q	u^*	σ^*	λ_b	ν	α
7	0.412	0.136	2.003	0.998	0.004
8	0.395	0.174	2.008	0.995	0.011
9	0.379	0.187	2.019	0.987	0.026
10	0.365	0.193	2.032	0.977	0.045

Table 3. Specific heat critical exponent (α): comparison between the RSRG result and the prediction from the Fisher renormalisation.

q	Pure	Fisher renormalisation	RSRG
7	0.010	-0.010	0.004
8	0.063	-0.067	0.011
9	0.107	-0.119	0.026
10	0.144	-0.168	0.045

6. Discussion

In this work the treatment of annealed dilute systems by RSRG methods has been investigated. The first approach (§ 2) provides a method in which, within a cluster approximation, the proper relationship between concentration and thermodynamic parameters is maintained while the bond variables are forced back to binary form. This approach is adequate for cases where no Fisher renormalisation should occur and is internally consistent provided the correct generalised hyperscaling relation (containing the fractal dimension) is used.

Such methods must however fail if randomness is a relevant perturbation and the remainder of the investigation is concerned with tests of this situation and, in particular, with providing a scheme which can account more fully for the associated evolution under scaling of the probability distribution of the annealed case. From this, flowlines were plotted for several integer values of q , and the flow was found to depend on the sign of the pure specific heat exponent in the way expected from the Fisher criterion. The random fixed points were obtained in the cases where α_{pure} was positive. The eigenvalues at the random fixed point were obtained from the linearised form of the transformation and from these, the random critical exponents.

Because the full constraint of the chemical potentials to give the temperature independent probability distribution was relaxed (in equation (26)), these exponents did not present numerical agreement with the Fisher renormalisation values. They nevertheless showed a change in the right direction (from the pure values towards the Fisher renormalisation ones). It would be desirable to allow more fully for the constraint in the chemical potentials, and it seems that for this it would be necessary to combine a technique like that used in § 2 (involving the relationship between p and the partition function) with the generalisation to non-binary bond variables which was introduced in the later sections.

We can again remark that the difficulties in handling the full distribution can be avoided and still have the Fisher renormalisation satisfied by performing the scaling of the equivalent pure system and then mapping the result back to the annealed one.

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